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On the stability of boundary-layer flow over a spring-mounted piston

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The stability of weakly compressible boundary-layer flow over a spring-mounted piston is examined theoretically by modelling the mean boundary layer at low Strouhal numbers by means of a step-function velocity profile. This constitutes a prototype problem for the treatment of the interaction of unsteady boundary-layer flow with a compliant surface, and the present discussion complements a recent analysis due to Ffowcs Williams and Purshouse by incorporating the influence of flow separation at the edges of the piston. This is effected analytically by application of the unsteady Kutta condition at both the leading and trailing edges of the piston. At high Reynolds numbers and in the case of light fluid loading it is predicted that the separated flow can cause piston flutter. Stability criteria are derived for a rectangular piston of large aspect ratio.

1. Introduction

Kramer (1957) has suggested that the use of compliant wall coatings of appropriate construction may lead to a significant reduction in turbulent boundary-layer drag. Such a reduction is likely to depend on the existence of a strong coupling between the wall motion and the turbulence. Flow/surface interactions are frequently 'noisy' (Curle 1955), however, and conditions that optimize the reduction in drag on, say, the fuselage of a wide-bodied jet transport, could well produce unacceptably high levels of aerodynamic sound. This issue is ignored in most of the recent studies of drag reduction (for an extensive bibliography see Frenkiel, Landahl & Lumley 1977).

The theoretical treatment of unsteady boundary-layer flow over a compliant surface is in practice too difficult to undertake in full generality. It is known (Benjamin 1960, 1963) that the basic instability of incompressible laminar flow over a rigid wall is not diminished when the surface is dissipative and compliant, and Howe (1979) has shown how a dissipative wall can attenuate incident sound while at the same time enhancing the intensity of the boundary-layer turbulence. Perhaps the simplest prototype model problem that involves simultaneously compliant surface/mean-flow/acoustic interactions is that discussed by Ffowcs Williams & Lovely (1975), Leppington & Levine (1979) and Levine (1980). These authors consider small oscillations of a circular spring-mounted piston whose face is flush with a plane rigid baffle in the undisturbed state. The stability and acoustic radiation were examined in the presence of an inviscid *uniform* mean flow. When the perturbed motion of the fluid is taken to be irrotational, Ffowcs Williams & Lovely demonstrated that the piston exhibits at most a *static divergence* instability, in which the piston is sucked into the flow until there is ultimate mechanical failure or equilibrium is established at a position determined by the nonlinear characteristic of the spring.

An airfoil in steady flight can experience an analogous instability, known as torsional divergence (Bisplinghoff & Ashley 1975, §6). But in practice 'flutter' instabilities, in which time-periodic oscillations grow steadily in amplitude, are more important in that they tend to occur at much lower velocities. They are absent in the potential-flow modelling of the piston problem because of the neglect of the mean shear of the boundary layer. Disturbances that are generated in the shear layer by the piston induce wall pressure fluctuations having a component in phase with the piston velocity and can therefore do work on the piston, resulting in large-amplitude oscillations and the emission of intense aerodynamic sound. A simple means of incorporating mean shear has been proposed by Ffowcs Williams & Purshouse (1981) in terms of a more general theory which draws an analogy between a real turbulent flow and an idealized inviscid model in which all of the mean shear is concentrated into a plane vortex sheet at a stand-off distance δ from the wall (which characterizes the thickness of the boundary-layer buffer zone). Small potential-flow perturbations are permitted on either side of the vortex sheet, whose motion in response to that of the piston and the boundary-layer turbulence can in principle involve the excitation of Kelvin–Helmholtz instability waves that grow indefinitely with distance downstream. In fact, such waves are deliberately excluded from the Ffowcs Williams–Purshouse theory: flutter instabilities are again found to be absent and they show that the static divergence of Ffowcs Williams & Lovely is possible provided that the radius of the piston exceeds about 5δ.

In this paper we shall argue on the basis of a linearized analysis that flutter can occur when the wake suppressed in the Ffowcs Williams–Purshouse theory is allowed to develop downstream of the piston. Linear perturbation theory predicts exponential growth of the wake with downstream distance and the detailed results cannot therefore be applicable for more than a few characteristic lengthscales from the piston, after which nonlinear processes will have evolved sufficiently to change the character of the motion. Nevertheless, the theory may still give an adequate representation of the *backreaction* of the wake on the piston. Indeed, careful observation (D. W. Bechert 1981, private communication) indicates that the motion in the vicinity of the trailing edge of a splitter plate separating flows of different mean velocities agrees well with the predictions of a linearized treatment of the unstable free shear layer in the wake of the plate. This suggests that the upstream influence on their source of shear-layer fluctuations in the nonlinear region is likely to be small. The implications of this hypothesis in the potential-flow analysis are discussed in §2.

The piston and boundary layer are strongly coupled at small values of the Strouhal number based on boundary-layer thickness. But contrary to expectation it turns out in this case (§2) that *no wake is formed* when the boundary layer is modelled by a linearly disturbed vortex sheet. This is a consequence of the inviscid potential-flow approximation. When the curvature of the piston is large, for example at the edge of a circular piston of top-hat profile, the action of viscosity in a high-Reynolds-number flow would tend to produce flow separation. This can be included formally in an inviscid analysis by application of an unsteady Kutta condition, as in thin airfoil theory (Ashley & Landahl 1965), and such a calculation is made in §3. The importance of viscosity was noted by Ffowcs Williams & Lovely (1975), who invoked it to limit the magnitude of potential-flow suction forces at sharp edges, but tacitly assumed that the corresponding contribution to the suction force had no component in phase with the velocity of the piston. The Kutta condition is particularly easy to apply when the boundary-layer thickness δ is small relative to all other pertinent lengthscales, and we consider two cases that, for mathematical convenience, assume the piston to be two-dimensional, or at least of high aspect ratio. In the first of these separation

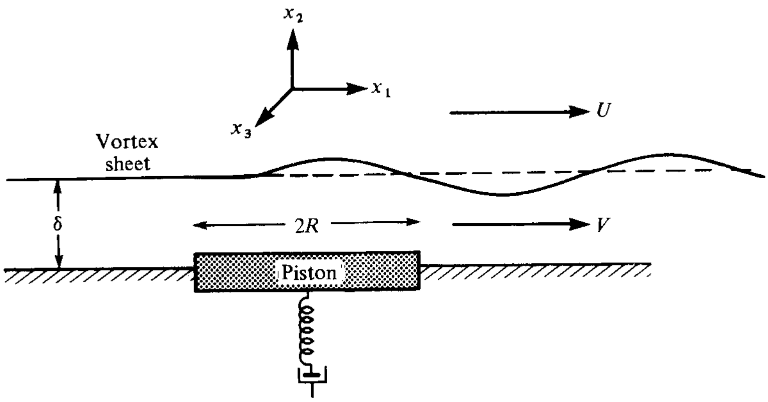

FIGURE 1. A circular piston of radius R executes small oscillations beneath a boundary layer whose mean-velocity profile is modelled by a step function.

occurs only at the trailing edge of the piston, and flutter instability is possible at finite aspect ratios when the reduced frequency ϵ, say, based on the width of the piston is small (§4). In the second case separation is postulated to occur at both the leading and trailing edges: instability is now possible at all reduced frequencies, although more likely when ϵ is small.

2. Potential-flow modelling of the instability problem

Consider small oscillations of a spring-mounted circular piston of radius R whose face is flush with an infinite plane baffle in the undisturbed state (see figure 1). The baffle lies in the plane $x_2 = 0$ of a rectangular coordinate system (x_1, x_2, x_3), with the origin at the equilibrium position of the centre of the piston. Parallel to the x_1 axis there exists a mean flow that has speed U in the free stream outside the boundary layer; the x_3 axis is directed out of the plane of the paper in the figure.

Let $h(t)$ denote the displacement of the piston in the positive x_2 direction from the equilibrium position. For small oscillations it is assumed that h satisfies the linear equation

$$m\frac{d^2h}{dt^2}+\beta\frac{dh}{dt}+Kh = -\mathscr{A}h, \tag{2.1}$$

in which m, β, K denote respectively the mass, internal-damping factor and the spring-stiffness coefficient of the piston. The net normal force exerted on the piston by the fluid is determined by the fluid-loading operator $\mathscr{A}$, which is defined in terms of the perturbation pressure $p(\mathbf{x}, t)$ by

$$\mathscr{A}h = \int_S p(x_1, 0, x_3, t)\, dx_1\, dx_3, \tag{2.2}$$

where, in the linearized approximation, the integration is taken over the region $S: |x_1^2+x_3^2|^{\frac{1}{2}} < R, x_2 = 0$.

The complex eigenfrequencies ω of the oscillations are the roots of the characteristic equation

$$m\omega^2+i\beta\omega-K = \mathscr{A}(\omega), \tag{2.3}$$

obtained by setting

$$h = h_0\, e^{-i\omega t} \tag{2.4}$$

in (2.1), where h_0 is constant. It may be assumed without loss of generality that $\mathrm{Re}(\omega) \geqslant 0$, and we can write

$$\mathscr{A}(\omega) = \mathscr{A}_{\mathrm{R}}(\omega) + i\mathscr{A}_{\mathrm{I}}(\omega), \tag{2.5}$$

where A_{R}, $\mathscr{A}_{\mathrm{I}}$ are real. Piston flutter occurs when the net effective damping is negative, i.e. when

$$\mathrm{Im}\,(\omega) > 0, \tag{2.6}$$

in which case oscillations of frequency $\mathrm{Re}\,(\omega)$ grow in amplitude in proportion to $\exp\{\mathrm{Im}\,(\omega)\,t\}$.

When both the fluid loading and internal damping of the piston are small, so that significant changes in amplitude occur only over many cycles of oscillation, the roots of (2.3) do not differ appreciably from $\pm\omega_0$, where

$$\omega_0 = \left(\frac{K}{m}\right)^{\frac{1}{2}}. \tag{2.7}$$

To leading order we have

$$\omega = -\frac{i}{m}\left(\beta - \frac{\mathscr{A}_{\mathrm{I}}(\omega_0)}{\omega_0}\right) + \left(\frac{K + \mathscr{A}_{\mathrm{R}}(\omega_0)}{m}\right)^{\frac{1}{2}}, \tag{2.8}$$

and condition (2.6) for the onset of fluttering becomes

$$\mathscr{A}_{\mathrm{I}}(\omega_0) > \beta\omega_0 > 0. \tag{2.9}$$

The static divergence discussed by Ffowcs Williams & Lovely (1975) corresponds to a root of (2.3) that is purely imaginary, and occurs when $\mathscr{A}$ is large enough to be comparable with the inertia $m\omega^2$ and stiffness K of the spring. This generally requires a large mean-flow velocity (see (2.19) below).

In the application of their general theory Ffowcs Williams & Purshouse (1981) consider the linearized problem in which the real boundary-layer flow over the piston is replaced by one in which the free stream at speed U is separated from the wall by a region of stagnant fluid of width δ. The mean shear is concentrated into a vortex sheet of strength U. The same model is adopted here, except that, as illustrated in figure 1, the velocity profile will be generalized to include a mean flow at speed $V(< U)$ in the x_1 direction between the wall and the vortex sheet. This is the velocity at which boundary-layer disturbances of wavelength $\gg \delta$ are convected downstream, and its inclusion provides an additional degree of freedom, which is useful in interpreting predictions of the theory. Modelling the boundary layer in this way should be an adequate first approximation at small Strouhal numbers $\omega\delta/U$.

When viscous stresses are neglected the perturbed motion of the fluid due to oscillations of the piston can be expressed in terms of a velocity potential $\phi e^{-i\omega t}$ on either side of the vortex sheet. According to linear theory, the boundary conditions that pressure and fluid particle displacement should be continuous across the vortex sheet may be applied at $x_2 = \delta$. The x_2 component of velocity at the wall ($x_2 = 0$) is non-zero only at the piston, where it equals the material derivative of the normal displacement, i.e. for the circular piston

$$\frac{\partial \phi}{\partial x_2} = -h_0\left(i\omega - V\frac{\partial}{\partial x_1}\right)H(R - |x|) \quad (x_2 = 0), \tag{2.10}$$

where the Heaviside step function $H(x) = 0, 1$ respectively as $x \lessgtr 0$. In $x_2 > \delta$ it is required that ϕ should be bounded at large distances from the wall.

The boundary condition (2.10) requires the normal velocity $v_n = \partial\phi/\partial x_2$ to have a δ-function singularity at the edge of the piston, which differs from the much weaker

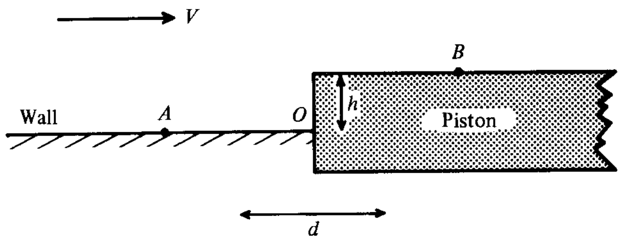

FIGURE 2. Illustrating the formulation of the boundary conditions for a sharp-edged piston.

algebraic singularity ($\sim x^{-\frac{1}{3}}$ at most, as $x \to 0$) that potential theory predicts for flow near a square edge. The use of (2.10) can be justified by means of the following argument.

Consider, for example, conditions at a leading edge, as illustrated in figure 2, and let the origin of coordinates be temporarily displaced to this position (point O in the figure). Inviscid theory requires that a fluid particle incident along the wall will also travel along the surface of the piston. Let A, B have x_1 coordinates $x_1 \mp \Delta$ respectively. The mean normal velocity between A and B is then

$$v_n = \frac{\zeta(x_1+\Delta, t+\tau)-\zeta(x_1-\Delta, t)}{\tau}, \tag{2.11}$$

where ζ is the normal displacement of the fluid particle (i.e. in the x_2 direction), and τ is its transit time from A to B. The distance d over which the mean flow is perturbed from its uniform velocity V must be of the same order as the displacement h of the piston from the wall. Hence, provided that $h \ll \Delta$, we can take $\tau = 2\Delta/V$, and

$$v_n = \frac{\partial \zeta}{\partial t}+\frac{V}{2\Delta}\{\zeta(x_1+\Delta, t)-\zeta(x_1-\Delta, t)\}. \tag{2.12}$$

The fractional departure of this expression from the exact value will be proportional to some non-negative power of the displacement h. Linear theory ignores such corrections, and the limit $\Delta \to 0$ then gives (2.10). According to this point of view, one is not interested in the detailed behaviour of the flow at the edge, but only in its influence at distances from the edge exceeding the piston displacement.

Following Ffowcs Williams & Purshouse (1981) we shall for the moment assume the flow to be *incompressible*, in which case the potential ϕ satisfies the Laplace equation $\nabla^2\phi = 0$ in $0 < x_2 < \delta$, $x_2 > \delta$. The boundary conditions are easily satisfied if ϕ is expressed as a Fourier integral, and in $0 < x_2 < \delta$ we find

$$\phi = -ih_0\int_{-\infty}^{\infty} \frac{f(\mathbf{k})(\omega-Vk_1)}{|\mathbf{k}|}\Bigg[\sinh(|\mathbf{k}|x_2) - \left(\frac{(\omega-Vk_1)^2\sinh(|\mathbf{k}|\delta)+(\omega-Uk_1)^2\cosh(|\mathbf{k}|\delta)}{(\omega-Vk_1)^2\cosh(|\mathbf{k}|\delta)+(\omega-Uk_1)^2\sinh(|\mathbf{k}|\delta)}\right)\cosh(|\mathbf{k}|x_2)\Bigg]e^{i\mathbf{k}\cdot\mathbf{x}}\,dk_1\,dk_3, \tag{2.13}$$

where $\mathbf{k} = (k_1, 0, k_3), \quad |\mathbf{k}| = (k_1^2+k_3^2)^{\frac{1}{2}}$.

Here

$$f(\mathbf{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} H(R-|\mathbf{x}|)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,dx_1\,dx_3 \quad (x_2 = 0)$$
$$= \frac{RJ_1(|\mathbf{k}|R)}{2\pi|\mathbf{k}|} \tag{2.14}$$

defines the Fourier transform of the piston profile in terms of the Bessel function J_1. In obtaining (2.13) it has been assumed that the density ρ_0, say, of the fluid is constant throughout the flow.

The solution (2.13) is substantially equivalent to that given by Ffowcs Williams & Purshouse (1981) when their turbulence volume source terms are discarded. It represents a flow that is bounded for all time, but it does not satisfy the causality condition that all disturbances of the boundary layer that are correlated with the motion of the piston arise as a consequence of that motion. Causality requires ϕ to be a regular function of ω for arbitrary large and positive values of Im (ω) (Titchmarsh 1948, chap. 1). The causal response of the flow for any given real or complex value of ω can therefore be obtained by analytic continuation of the solution (2.13) from Im $(\omega) = +i\infty$. Any singularities of the integrand that cross the real k_1, k_3 axes during this operation are to be accommodated by an appropriate deformation of the integration contours.

The perturbation pressure $p(\mathbf{x}, t) = \bar{p}(\mathbf{x})\, e^{-i\omega t}$, say, is given in $0 < x_2 < \delta$ by the linearized Bernoulli equation

$$\bar{p} = \rho_0\left(i\omega - V\frac{\partial}{\partial x_1}\right)\phi, \tag{2.15}$$

and this may be used to calculate the fluid-loading operator by making use of the definition (2.2):

$$\mathscr{A}(\omega)\, h_0 = \rho_0 \int_S \left(i\omega - V\frac{\partial}{\partial x_1}\right)\phi(x_1, 0, x_3)\, dx_1\, dx_3. \tag{2.16}$$

Using (2.13), we obtain the formal representation

$$\mathscr{A}(\omega) = -4\pi^2\rho_0 \iint_{-\infty}^{\infty} \frac{|f(\mathbf{k})|^2(\omega - Vk_1)^2}{|\mathbf{k}|} \times \left(\frac{(\omega - Vk_1)^2 \sinh(|\mathbf{k}|\delta) + (\omega - Uk_1)^2\cosh(|\mathbf{k}|\delta)}{(\omega - Vk_1)^2\cosh(|\mathbf{k}|\delta) + (\omega - Uk_1)^2 \sinh(|\mathbf{k}|\delta)}\right) dk_1\, dk_3. \tag{2.17}$$

Actually, this formula is generally valid only for pistons having *smooth* profiles. The discontinuity h_0 in fluid-particle displacement at the edge of a piston of top-hat profile implies that $f(\mathbf{k})$, defined in (2.14), can decay no faster than $|\mathbf{k}|^{-\frac{3}{2}}$ as $|\mathbf{k}| \to \infty$. It then follows from (2.17) that $\mathscr{A}(\omega)$ is *unbounded* except when $V = 0$, the case treated by Ffowcs Williams & Purshouse (1981), and we shall discuss this first.

When $\omega\delta/U \gg 1$ (and $V = 0$) Ffowcs Williams & Purshouse obtained the leading approximation to (2.17) (the integration being confined to the real k_1, k_3 axes) in the form

$$\mathscr{A}(\omega) = -\tfrac{8}{3}\rho_0 R^3\omega^2 + \rho_0 U^2 RX. \tag{2.18}$$

The first term on the right of this result represents the additional inertia of the motion due to the fluid displaced by the piston. The second is independent of ω, and corresponds to a modification of the effective spring stiffness by the flow. X is a dimensionless real coefficient, which depends on the ratio R/δ. Substitution of (2.18) into (2.3) gives the characteristic frequency

$$\omega = +\left(\frac{K + \rho_0 U^2 RX}{m + \frac{8}{3}\rho_0 R^3}\right)^{\frac{1}{2}}, \tag{2.19}$$

when the internal damping of the piston is neglected. Provided that $X > 0$ it follows that the oscillations of the piston are stable. Ffowcs Williams & Purshouse show that X is negative for $R/\delta \geqslant 5$, and a sufficiently large mean-flow velocity U can then induce a divergence instability of the type predicted by Ffowcs Williams & Lovely (1975).

These conclusions are, however, based on an evaluation of (2.17) that deliberately suppresses any contribution from singularities of the integrand that are encountered during application of the causality condition by analytic continuation in the ω-plane. To examine the influence of causality we shall assume the fluid loading to be small enough that the characteristic frequencies are well approximated by (2.8), in which $\mathscr{A}(\omega)$ is to be evaluated at the real frequency ω_0. Taking $\omega = \omega_0$ in (2.17), we see that the integral along the real k_1, k_3 axes is *real* and does not affect the stability, since it makes no contribution to the imaginary part $\mathscr{A}_{\mathrm{I}}(\omega_0)$ of $\mathscr{A}$. When $|\omega|$ is large (relative to U/δ) the integrand in (2.17) has simple poles at

$$k_1 = \frac{\omega(1 \pm i)}{U}, \tag{2.20}$$

which lie in the upper half of the k_1 plane if $\frac{1}{4}\pi < \arg\omega < \frac{3}{4}\pi$. As ω moves from this sector to ω_0 (> 0) one of these poles crosses the positive real k_1 axis to the point $\kappa = \omega_0(1-i)/U$, and therefore makes a residue contribution to $\mathscr{A}(\omega_0)$ given by

$$\mathscr{A}_\kappa = -i(2\pi)^3\rho_0 \int_{-\infty}^{\infty} \left(\frac{|f(\mathbf{k})|^2 \omega_0^2}{|\mathbf{k}|} \times \frac{\{\omega_0^2 \sinh(|\mathbf{k}|\delta) + (\omega_0 - Uk_1)^2 \cosh(|\mathbf{k}|\delta)\}}{\dfrac{\partial}{\partial k_1}\{\omega^2{}_0 \cosh(|\mathbf{k}|\delta) + (\omega_0 - Uk_1)^2 \sinh(|\mathbf{k}|\delta)\}} \right)_{k_1=\kappa} dk_3. \tag{2.21}$$

This is valid only for $\kappa\delta \gg 1$, when the pole can be identified with a Kelvin–Helmholtz instability wave of a free shear layer proportional to $\exp(i\kappa x_1)$, whose amplitude (according to linear theory) increases indefinitely with distance downstream of the piston. Since $|\mathbf{k}|\delta = (k_3^2+\kappa^2)^{\frac{1}{2}}\delta \gg 1$, the remaining integration in (2.21) can be evaluated asymptotically. For example, in the case of small reduced frequencies $\omega_0 R/U$ we obtain

$$\mathscr{A}_\kappa \simeq -\frac{\pi\rho_0 R^4 \omega_0^3}{U}\left(\frac{\pi}{\kappa\delta}\right)^{\frac{1}{2}} e^{-2\kappa\delta}. \tag{2.22}$$

The imaginary part of this expression is alternately positive and negative in frequency intervals of width $\pi U/2\delta$, which indicates (cf. (2.9)) that the necessary condition $\mathscr{A}_{\mathrm{I}} > 0$ for the occurrence of flutter can be satisfied, although flutter is probably unlikely in practice because $\mathscr{A}_\kappa$ is exponentially small.

Ffowcs Williams & Purshouse (1981) emphasized that the strongest interaction between the piston and shear layer must occur at small values of the Strouhal number $\omega_0\delta/U$. Similarly, fluttering is likely to occur only if the reduced frequency $\omega_0 R/U$ is not too large, since otherwise phase cancellation of pressure fluctuations on the surface of the piston will produce very small values of $\mathscr{A}_{\mathrm{I}}(\omega_0)$. But according to (2.17), in the extreme in which $\delta \to 0$, so that δ is small compared with all other length scales, we have for *arbitrary* $V \geqslant 0$

$$\mathscr{A}(\omega) = -4\pi^2\rho_0 \iint_{-\infty}^{\infty} \frac{(\omega - Uk_1)^2|f(\mathbf{k})|^2}{|\mathbf{k}|} dk_1\, dk_3. \tag{2.23}$$

This is unbounded for the sharp-edged top-hat piston, irrespective of the value of V. The integral can always be made finite, however, by suitably 'rounding' the edge of the piston, and the top-hat case may be regarded as the limit of a sequence of such rounded profiles. Evidently $\mathscr{A}_{\mathrm{I}} = 0$ for real $\omega = \omega_0$ and each member of the sequence, so that, although $\mathscr{A}_{\mathrm{R}}$ is ultimately unbounded, in the approximation of small fluid loading the motion is stable. This is a situation in which the piston has a significant influence on the boundary-layer motion in its immediate vicinity, but, surprisingly, the theory predicts that *no wake is generated*. Indeed, if $\zeta e^{-i\omega t}$ denotes the x_2 component of displacement of the vortex sheet, the representation (2.13) of ϕ in $0 < x_2 < \delta$ can be used together with the kinematical relation

$$\frac{\partial \phi}{\partial x_2} = -\left(i\omega - V\frac{\partial}{\partial x_1}\right)\xi \quad (x_2 = \delta) \tag{2.24}$$

to show that, as $\delta \to 0$,

$$\begin{aligned}\zeta &\to h_0 \iint_{-\infty}^{\infty} f(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, dk_1\, dk_3 \quad (x_2 = 0)\\ &= h_0 H(R - |\mathbf{x}|),\end{aligned} \tag{2.25}$$

i.e. the displacement of the vortex sheet ultimately mimics exactly that of the piston.

This physically unrealistic behaviour is a consequence of the potential-flow approximation to the perturbed motion, in which the streamlines (and therefore the vortex sheet when $\omega_0 \delta / U \to 0$) are required to turn through a right-angle at the edge $(x_1^2 + x_3^2)^{\frac{1}{2}} = R$ of the piston. At such points the pressure is infinite. Ffowcs Williams & Lovely (1975) argue that the value of $\mathscr{A}_{\mathrm{R}}$ would be held finite in practice by the action of viscosity at the edges of the piston, while $\mathscr{A}_{\mathrm{I}}$ would be unchanged from its inviscid value. The proposal is unsatisfactory, however, in that it implies that the modification of $\mathscr{A}$ due to viscosity is always in phase with the displacement of the piston. Except in very-low-Reynolds-number flows, viscosity must cause flow separation at the edges of the piston. This will affect both the real and imaginary parts of $\mathscr{A}(\omega_0)$, since vorticity generated in this manner will produce variations in the fluid loading whose phase is dependent on the velocity at which the vorticity is swept downstream in the mean flow. The importance of this phenomenon will now be considered.

3. The influence of separation at low Strouhal numbers in a weakly compressible fluid

We shall simplify the analysis by considering a two-dimensional flat-faced piston that in equilibrium occupies the region $|x_1| < s$, $x_2 = 0$, $-\infty < x_3 < \infty$. The thickness δ of the model boundary layer in figure 3 is assumed to be small compared with all other characteristic lengths except the displacement amplitude h_0 of the piston (i.e. $\omega\delta/U \ll 1$), and in these circumstances we anticipate a strong interaction between the piston and boundary layer. In the absence of separation the displacement of the vortex sheet is given by

$$\zeta = h_0 H(s - |x_1|), \tag{3.1}$$

which is the two-dimensional analogue of (2.25). The effect of separation at the edges of the piston is to remove the discontinuities in ζ that occur at $x_1 = \pm s$.

Separation can be modelled analytically by the introduction of sources at the edges of the piston. At the trailing edge, for example, we insert a line singularity in normal velocity whose strength is chosen to ensure that ζ varies continuously there. In order to do this it is first necessary to solve the auxiliary problem for the potential ϕ_{S}, say, which satisfies

$$\frac{\partial \phi_{\mathrm{S}}}{\partial x_2} = A\delta(x_1 - s) \quad \text{on} \quad x_2 = 0, \tag{3.2}$$

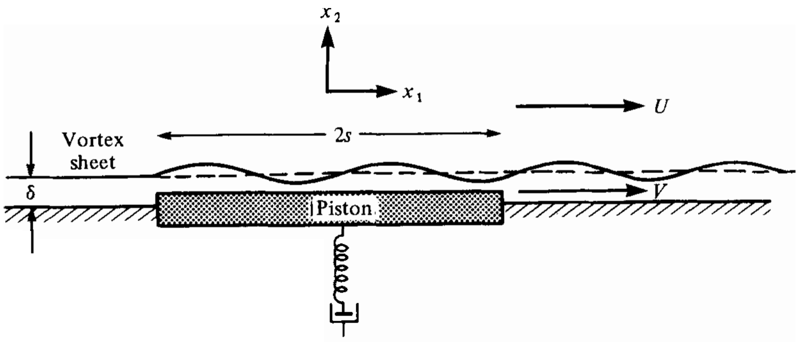

FIGURE 3. A two-dimensional piston of width $2s$ beneath a thin boundary layer, which, for $\omega\delta/U \ll 1$, is modelled by a step-function velocity profile.

where A is the source strength. In $x_2 > \delta$ we find as $\omega\delta/U \to 0$, and for *incompressible* flow

$$\phi_s = \frac{-A}{2\pi}\int_{-\infty}^{\infty}\frac{\omega-Uk}{\omega-Vk}\frac{1}{|k|}\exp\left[ik(x_1-s)-|k|(x_2-\delta)\right]dk, \tag{3.3}$$

in which causality requires the integration contour to pass below the pole at $k = \omega/V$. The integral exists in the sense of a generalized function, and provides a definition of ϕ_s that is unique to within an additive arbitrary constant, since $1/|k|$ is defined only to within an arbitrary multiple of $\delta(k)$ (Lighthill 1958, p. 43). The corresponding displacement ζ_s of the vortex sheet is obtained from (2.24) (in which V is replaced by U in $x_2 > \delta$):

$$\zeta_s = \frac{iA}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x_1-s)}}{\omega-Vk}dk. \tag{3.4}$$

Since $\operatorname{Im}(\omega) > 0$, we have

$$\zeta_s = \frac{A}{V}H(x_1-s)\exp\left[i\frac{\omega}{V}(x_1-s)\right], \tag{3.5}$$

i.e. the vortex sheet is perturbed only in the region downstream of the source, where its displacement fluctuations propagate as a wave at speed V. Combining this displacement with that given in (3.1) it follows that, if the source strength $A = Vh_0$, the net displacement of the vortex sheet becomes

$$\zeta = h_0\left\{H(s-|x_1|)+H(x_1-s)\exp\left[i\frac{\omega}{V}(x_1-s)\right]\right\} \tag{3.6}$$

(*case I*), which is continuous at the trailing edge $x_1 = s$ of the piston.

The same procedure is used to eliminate the discontinuity at the leading edge, and gives the following composite representation of ζ (again for incompressible flow):

$$\zeta = h_0\left\{H(s-|x_1|)+H(x_1+s)\exp\left[i\frac{\omega}{V}(x_1+s)\right]+H(x_1-s)\exp\left[i\frac{\omega}{V}(x_1-s)\right]\right\} \tag{3.7}$$

(*case II*), which is continuous everywhere.

The influence of separation on the stability of the piston will be examined in each of cases I, II. In case I separation occurs at the trailing edge only, so that the real part $\mathscr{A}_R$ of the fluid-loading operator will be unbounded because of the remaining singularity at the leading edge. As in §2, we shall formally assume that $\mathscr{A}_R$ is maintained at some finite value by viscous action and/or local rounding of the leading edge of the piston (on a lengthscale that is small relative to h_0) in a manner that does not affect the value of $\mathscr{A}_I$.

The calculation of the fluid-loading operator $\mathscr{A}(\omega)$ by integration of the surface pressure over the piston is simplified in the low-Strouhal-number limit by the fact that the variation of the perturbation pressure across the boundary layer can be neglected. Thus the value of p just above the vortex sheet ($x_2 = \delta + 0$) may be used, *and this has the added advantage of permitting account to be taken of weak compressibility of the fluid* in a relatively straightforward manner. To do this observe that in $x_2 > \delta$ the potential can be expressed in the form

$$\phi(\mathbf{x}) = \int_{-\infty}^{\infty} G(\mathbf{x}, y_1) \frac{\partial \phi(\mathbf{y})}{\partial y_2} dy_1, \tag{3.8}$$

where the integration is taken along the mean position $y_2 = \delta + 0$ of the vortex sheet, and $G(\mathbf{x}, y_1)$ is a Green function that satisfies in $x_2 > \delta$ the convected wave equation

$$\left\{\frac{1}{c^2}\left(-i\omega + U\frac{\partial}{\partial x_1}\right)^2 - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\right\} G(\mathbf{x}, y_1) = 0$$

with

$$\frac{\partial G(\mathbf{x}, y_1)}{\partial x_2} = \delta(x_1 - y_1) \quad \text{on} \quad x_2 = \delta + 0. \tag{3.9}$$

Here c denotes the speed of sound, and the condition of weak compressibility requires that the acoustic wavelength $\sim 2\pi c/\omega \gg 2s$, the width of the piston. We consider only flows of small mean-flow Mach number $M = U/c$, in which case the hydrodynamic disturbances on the vortex sheet have lengthscale $2\pi V/\omega \ll$ acoustic wavelength, and this justifies our earlier neglect of compressibility in calculating the effective edge sources responsible for separation.

The piston is in the near field of the sound produced by its motion and interaction with the wake. It follows that in calculating the surface pressure from (3.8) it is sufficient to use the near-field approximation to $G(\mathbf{x}, y_1)$. This has the general form

$$G(\mathbf{x}, y_1) = \frac{1}{\pi} \ln\{[(x_1 - y_1)^2 + (x_2 - \delta)^2]^{\frac{1}{2}}/s\} + C, \tag{3.10}$$

where C is a complex constant whose value is determined by boundary conditions far from the piston (examples are discussed below in §4). For example, C assumes different values in the two cases in which (i) the mean flow extends to $x_2 = +\infty$ above the wall, and (ii) the wall containing the piston constitutes a sidewall of a duct carrying the mean flow.

3.1. *Calculation of the fluid loading in case I*

At $x_2 = \delta + 0$, just above the vortex sheet, we have $\partial\phi/\partial x_2 = -(i\omega - U\partial/\partial x_1)\zeta$, the material derivative of the displacement. It follows from (3.8) and the linearized Bernoulli equation (2.15), with V replaced by U, that in $x_2 > \delta$

$$\bar{p}(\mathbf{x}) = -\rho_0 \left(\omega + iU\frac{\partial}{\partial x_1}\right)^2 \int_{-\infty}^{\infty} \zeta(y_1)\, G(\mathbf{x}, y_1)\, dy_1. \tag{3.11}$$

As $\omega\delta/U \to 0$ this expression will also give the perturbation pressure on the surface of the piston when $|x_1| < s, x_2 = \delta$.

Introduce the reduced frequencies

$$\epsilon = \frac{\omega_0 s}{U}, \quad \sigma = \frac{\omega_0 s}{V} = \frac{\epsilon U}{V}. \tag{3.12}$$

Using (3.6) we can write in case I

$$\bar{p}(\xi) = \frac{h_0 \rho_0 U^2}{s}\left(\epsilon + i\frac{\partial}{\partial \xi}\right)^2 \int_{-1}^{\infty} G(\xi,\eta)\{H(1-|\eta|)+H(\eta-1)\,e^{i(\eta-1)\sigma}\}d\eta \quad (x_2 = \delta), \tag{3.13}$$

where

$$G(\xi,\eta) = \frac{1}{\pi}\ln|\xi-\eta|+C, \tag{3.14a}$$

$$\xi = \frac{x_1}{s}, \quad \eta = \frac{y_1}{s}. \tag{3.14b, c}$$

The causality condition is invoked to ensure convergence of the infinite integral in (3.13); the condition $\operatorname{Im}(\omega) > 0$ implies that the integrand is exponentially small as $\eta \to \infty$.

The contribution to the surface pressure from the logarithm in (3.13) is singular at the leading edge $\xi = -1$ of the piston because of the absence of separation (or other real fluid effects) at that point. Extracting the singularity of the integral, one finds that in the neighbourhood of $\xi = -1$

$$\bar{p} \approx -\frac{h_0 \rho_0 U^2}{\pi s}\,\mathsf{P}\left(\frac{1}{1+\xi}\right), \tag{3.15}$$

where P denotes *principal value*. This is a real quantity that formally makes no contribution to $\mathscr{A}_{\mathrm{I}}(\omega_0)$. The latter is determined by the imaginary part of $\bar{p}(\xi)$, and its value (*per unit length* in the spanwise (x_3) direction) is predicted by (2.2), (3.13) to be

$$\mathscr{A}_{\mathrm{I}}(\omega_0) = \rho_0 U^2\left\{4\epsilon^2 \operatorname{Im}(C)+\frac{\epsilon^2}{\sigma}[2\operatorname{Re}(C)+\mathscr{F}(\epsilon)]\right\}, \tag{3.16}$$

where

$$\mathscr{F}(\epsilon) = \frac{1}{\pi}\left[2(\ln 2-1)+\frac{(1-\sigma/\epsilon)^2}{\sigma}(\tfrac{1}{2}\pi+\operatorname{si}(2\sigma)\cos 2\sigma-\operatorname{ci}(2\sigma)\sin 2\sigma)\right], \tag{3.17}$$

in which $\operatorname{si}(x)$, $\operatorname{ci}(x)$ denote respectively sine and cosine integrals defined for $x > 0$ by (Gradshteyn & Ryzhik 1980, p. 928)

$$\operatorname{si}(x) = -\int_x^{\infty}\frac{\sin t}{t}dt, \quad \operatorname{ci}(x) = -\int_x^{\infty}\frac{\cos t\,dt}{t}. \tag{3.18}$$

The terms in the brace brackets of (3.16) that involve σ arise solely from the presence of the wake, i.e. as a result of trailing-edge separation. The first term in the braces represents the influence of compressibility, viz of *radiation damping* (since in practice $\operatorname{Im}(C) < 0$).

3.2. *The fluid-loading operator in case II*

In this case $\mathscr{A}(\omega)$ is bounded because the pressure is finite at both the leading and trailing edges of the piston. Making use of integrals tabulated by Gradshteyn & Ryzhik (1980) we find

$$\mathscr{A}(\omega_0) = \rho_0 U^2\left\{\frac{2\epsilon^2}{\pi}(2\ln 2-3)+4\epsilon^2 C + \frac{2(\epsilon-\sigma)^2}{\sigma^2}[\operatorname{si}(2\sigma)\sin 2\sigma+\operatorname{ci}(2\sigma)\cos 2\sigma-\gamma-\ln 2\sigma]+\frac{i(\epsilon-\sigma)^2}{\sigma^2}[1-e^{2i\sigma}]\right\}, \tag{3.19}$$

in which $\gamma = 0{\cdot}577216\ldots$ is Euler's constant. The stability characteristics are defined by $\mathscr{A}_{\mathrm{I}}(\omega_0)$:

$$\mathscr{A}_{\mathrm{I}}(\omega_0) = 2\rho_0 U^2 \left\{ 2\epsilon^2 \,\mathrm{Im}\,(C) + (\epsilon-\sigma)^2 \frac{\sin^2\sigma}{\sigma^2} \right\}, \tag{3.20}$$

where, as before, the first term in the braces is negative and accounts for the radiation damping, and the second ($\geqslant 0$) denotes the destabilizing effect of separation.

4. The stability of pistons of large and infinite aspect ratios

4.1. *Case I: trailing-edge separation only*

The results of §3 are now used to examine the stability of a piston in an infinite plane baffle when the mean flow extends to $x_2 = +\infty$. On $x_2 = \delta$ the Green function in the limit of infinitesimal mean-flow Mach number is given by

$$G(x_1, \delta, y_1) = -\tfrac{1}{2} i H_0^{(1)} \left\{ \frac{\omega}{c} |x_1 - y_1| \right\}, \tag{4.1}$$

where $H_0^{(1)}(x)$ is a Hankel function of the first kind. When x_1, y_1 are in the neighbourhood of a piston of compact chord the argument of the Hankel function is small, and to leading order we can write, as in (3.14),

$$G(\xi, \eta) = \frac{1}{\pi} \ln|\xi - \eta| - \tfrac{1}{2} i + \frac{1}{\pi}\left[\gamma + \ln\frac{\omega s}{2c}\right]. \tag{4.2}$$

The compactness condition $\omega_0 s/c \ll 1$ implies that the real part of the constant term on the right of (4.2) is negative (i.e. $\mathrm{Re}\,(C) < 0$) and $\rightarrow -\infty$ as the sound speed $c \rightarrow \infty$ (limit of incompressible flow). In this case it follows from (3.16), (3.17) that $\mathscr{A}_{\mathrm{I}}(\omega_0)$ is negative for arbitrary values of ϵ, σ, and therefore, according to (2.8), that the motion is always stable.

It might be expected that the results of §3 would also yield the stability criterion of a piston of large, but finite, aspect ratio $l/2s$, l being the spanwise length of the piston, provided that the coefficient C in (3.14) could be assigned an appropriate value. This would be the case if $\omega_0 l/V \gg 1$, i.e. the hydrodynamic wavelength of the separated flow is small relative to l, so that the motion is essentially two-dimensional except in the immediate neighbourhoods of the side edges of the piston. It can be shown (Howe 1981*a*) that C should in fact take the value

$$C = \frac{1}{\pi} \ln \frac{e}{4\mathscr{R}} - \frac{i\omega_0 l}{2\pi c}, \tag{4.3}$$

where $\mathscr{R} = l/2s$ denotes the aspect ratio, and $e = 2{\cdot}71828\ldots$ is the base of the natural logarithm. The compactness requirement is that $\omega_0 l/c \ll 1$, so that $\mathrm{Im}\,(C) \rightarrow 0$ as $c \rightarrow \infty$. A necessary condition for *instability* of the piston is $\mathscr{A}_{\mathrm{I}}(\omega_0) > 0$. Discarding the small imaginary term in (4.3), this becomes, according to (3.16):

$$\frac{2}{\pi} \ln \frac{e}{4\mathscr{R}} + \mathscr{F}(\epsilon) > 0. \tag{4.4}$$

The left-hand side of this inequality is negative except for small values of ϵ. The variation of $\mathscr{F}(\epsilon)$ with the reduced frequency $\epsilon = \omega_0 s/U$ depends on the value of ϵ/σ, which, in the vortex sheet model, is equal to V/U, i.e. to the ratio of the phase velocity of boundary-layer disturbances in the wake to the free-stream velocity.

Figure 4 illustrates the stability boundaries defined by (4.4) in terms of the aspect

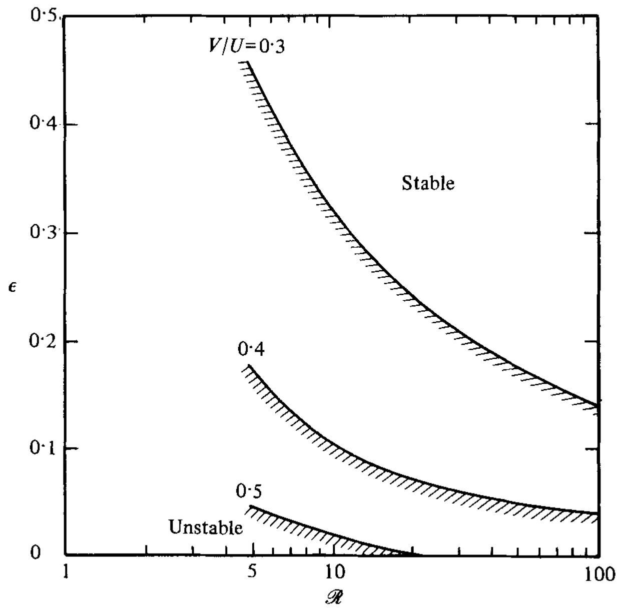

FIGURE 4. Stability boundaries for a rectangular piston of large aspect ratio $\mathscr{R} = l/2s$ when separation occurs at the trailing edge.

ratio and the reduced frequency ϵ. The strongest tendency to instability is associated with low-convection-velocity, low-Strouhal-number boundary-layer waves and pistons of small aspect ratio, although the present asymptotic theory is unlikely to remain valid for $\mathscr{R} \lesssim 5$.

4.2. *Case II: separation at the leading and trailing edges*

The use of (3.20), (4.2) implies that for a piston of infinite aspect ratio the necessary condition for instability, $\mathscr{A}_{\mathrm{I}} > 0$, becomes

$$\left(1-\frac{\sigma}{\epsilon}\right)^2 \frac{\sin^2\sigma}{\sigma^2} > 1. \tag{4.5}$$

This inequality can always be satisfied for sufficiently small reduced frequency ϵ (i.e. sufficiently high mean-flow velocity), provided only that $\epsilon/\sigma = V/U < 0{\cdot}5$.

For an acoustically compact piston of spanwise dimension l, $\mathrm{Im}\,(C)$ given by (4.3) can normally be neglected, in which case (3.20) reveals that $\mathscr{A}_{\mathrm{I}} \geqslant 0$ for $\epsilon \neq \sigma$, so that the necessary condition for instability is always fulfilled. The absolute stability criterion follows from (2.9), which involves the internal damping of the piston. Recalling that $\mathscr{A}_{\mathrm{I}}$ defined in (3.20) refers to the fluid loading *per unit span*, we find that, when the small contribution from radiation damping is ignored, the motion will be *unstable* if

$$\frac{\beta}{2\rho_0 Usl} < \epsilon\left(1-\frac{\sigma}{\epsilon}\right)^2 \frac{\sin^2\sigma}{\sigma^2}. \tag{4.6}$$

The corresponding stability boundaries for three different values of ϵ/σ, the fractional phase velocity of the boundary-layer disturbances, are shown in figure 5. These results demonstrate, again, how the region of instability rapidly increases with diminishing phase velocity.

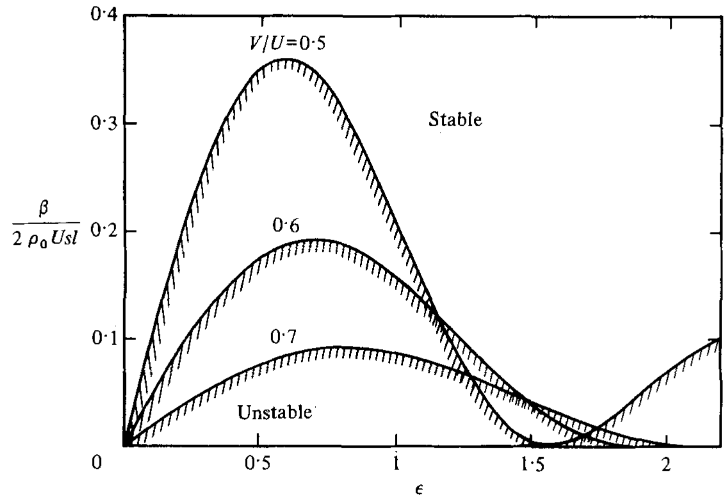

FIGURE 5. Stability boundaries for a rectangular piston of large aspect ratio $l/2s$ when separation occurs at both the leading and trailing edges.

5. Conclusion

The stability of boundary-layer flow over a spring-loaded piston in a wall has been examined by modelling the mean boundary layer by means of a step-function velocity profile. This is expected to be an adequate approximation at small values of the Strouhal number $\omega_0 \delta/U$. For an ideal fluid in which the mean shear is ignored, the work of Ffowcs Williams & Lovely (1975) has shown that the piston can exhibit only a *static divergence* type of instability. In the presence of a mean-velocity profile the piston generates boundary-layer waves in its wake whose backreaction causes it to flutter. The effect tends to be very weak, however, if viscosity is neglected in calculating the motion of the piston, even though disturbances created in the wake grow, according to linear theory, exponentially with distance downstream of the piston. The action of viscosity at the edges of the piston can result in surface-pressure fluctuations that are in phase with the piston velocity, and thereby lead to a net transfer of energy from the mean to the perturbation motion.

In particular, viscosity causes separation at the edges of a piston that executes low-Strouhal-number oscillations beneath a high-Reynolds-number boundary layer. This has been modelled theoretically, as in thin-airfoil theory, by an application of the unsteady Kutta condition at the edges. By this means it has been shown that rectangular pistons of large aspect ratio exhibit various regimes of flutter instability. Such pistons were considered for mathematical convenience, but the generality of the conclusions at small reduced frequencies (based on the streamwise dimension of the piston) are not expected to be significantly dependent on piston geometry. Two models have been examined. The first (case I) involves unsteady separation only at the trailing edge of the piston, while separation is permitted at both the leading and trailing edges in case II. In thin-airfoil theory it is usual to invoke the Kutta condition at a trailing edge, although there is no reason to exclude the possibility of unsteady separation at a *sharp* leading edge (cf. Howe 1981*b*). In the piston problem it seems unreasonable to demand that separation occur only at the trailing edge, so that our analytical conclusions for case II are more likely to be relevant in practice. These are that flutter instability is always possible in principle, for sufficiently small internal dumping of the piston, but more likely at small reduced frequencies.

The larger part of the work reported here was undertaken while the author was at Bolt Beranek and Newman, Inc., Cambridge, Mass.

REFERENCES

ASHLEY, H. & LANDAHL, M. 1965 *Aerodynamics of Wings and Bodies*. Addison-Wesley.

BENJAMIN, T. B. 1960 Effects of a flexible boundary on hydrodynamic stability. *J. Fluid Mech.* **9**, 513–532.

BENJAMIN, T. B. 1963 The threefold classification of unstable disturbances in flexible surfaces bounding inviscid flows. *J. Fluid Mech.* **16**, 436–450.

BISPLINGHOFF, R. L. & ASHLEY, H. 1975 *Principles of Aeroelasticity*. Dover.

CURLE, N. 1955 The influence of solid boundaries upon aerodynamic sound. *Proc. R. Soc. Lond.* A **231**, 505–514.

FFOWCS WILLIAMS, J. E. & LOVELY, D. J. 1975 Sound radiation into uniformly flowing fluid by compact surface vibration. *J. Fluid Mech.* **71**, 689–700.

FFOWCS WILLIAMS, J. E. & PURSHOUSE, M. 1981 A vortex sheet modelling of boundary-layer noise. *J. Fluid Mech.* **113**, 187–220.

FRENKIEL, F. N., LANDAHL, M. T. & LUMLEY, J. L. (eds) 1977 *Proc. IUTAM Symp. on Structure of Turbulence and Drag Reduction, Washington, D.C., June* 1976. *Phys. Fluids Suppl.* **20**, S1–S297.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1980 *Table of Integrals, Series and Products* (corrected edn). Academic.

HOWE, M. S. 1979 A note on the interaction of unsteady flow with an acoustic liner. *J. Sound Vib.* **63**, 429–436.

HOWE, M. S. 1981*a* The influence of mean shear on unsteady aperture flow, with application to acoustical diffraction and self-sustained cavity oscillations. *J. Fluid Mech.* **109**, 125–146.

HOWE, M. S. 1981*b* The role of displacement thickness fluctuations in hydroacoustics. *Proc. R. Soc. Lond.* A **374**, 543–568.

KRAMER, M. O. 1957 Boundary layer stabilisation by distributed damping. *J. Aero. Sci.* **24**, 459–463.

LEPPINGTON, F. G. & LEVINE, H. 1979 The effect of flow on the piston problem of acoustics. *J. Sound Vib.* **62**, 3–17.

LEVINE, H. 1980 A note on sound radiation into a uniformly flowing medium. *J. Sound Vib.* **71**, 1–8.

LIGHTHILL, M. J. 1958 *Fourier Analysis and Generalised Functions*. Cambridge University Press.

TITCHMARSH, E. C. 1948 *Theory of Fourier Integrals*, 2nd edn. Oxford University Press.

WILLMARTH, W. W. 1975 Pressure fluctuations beneath turbulent boundary layers. *Ann. Rev. Fluid Mech.* **7**, 13–38.